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# CANONICAL FORM OF AN ELASTOPLASTIC MODEL OF NUCLEAR FUSION

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Starting from equations of motion describing the fusion process in symmetrical nuclear systems of low angular momenta we reconstruct the collective Lagrangian and dissipation Rayleigh functions. This opens new perspectives in studying the dynamical effects in the heavy nuclei collisions. In particular, it provides a basis for a quantal description of the fusion process and accompanying it effects.

The investigation has been performed at the Bogoliubov Laboratory of Theoretical Physics and Laboratory of Nuclear Problems, JINR.

## Каноническая форма эластопластической модели слияния ядер

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Для уравнений движения, описывающих процесс слияния в симметричной ядерной системе при низких угловых моментах, восстанавливается коллективный лагранжиан и диссипативная функция Рэлея. Это открывает новые возможности в изучении динамических эффектов в столкновениях тяжелых ионов, а также дает основу для квантового описания процесса слияния и сопровождающих его эффектов.

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## 1. Introduction

In Refs. 1, 2, 3 we had shown how to bring to the canonical form the equations of motion of a simple elastoplastic system resembling the one used to describe the fusion of identical nuclei after the central collision [4]. Giving a canonical form to the equations of motion has brought to evidence quite unexpected features of the fusion process related with thermal fluctuations. However, the description of new phenomena presented in Refs. 1, 3 has a qualitative character due to the simplifications introduced into the equations of motion. To arrive at the quantitative analysis of these phenomena and to go further in the description of fusion one must learn how to formulate in a canonical way the more complicated equations of Ref. 4. This is precisely the subject of this paper.

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## 2. An Elastoplastic Model of Fusion

Consider a system described by the equations [4]

$$\frac{1}{2}\ddot{Q} - \kappa(Q)\,(\dot{Q})^2 + W_{2,0}(Q) - \Pi(t) = 0, \tag{1}$$

$$\dot{\Pi} + \dot{Q}F_{fs}(Q) = I_{\rm rel}. \tag{2}$$

Here $Q$ is the macroscopic variable associated with the shape of the system (the mass-quadrupole moment), while $\Pi$ is another macroscopic variable describing the distribution of fermions in the momentum space (momentum-quadrupole moment). The right-hand side of Eq.(2) describes the effects of collective energy dissipation. We assume the following expression for $I_{\rm rel}$:

$$I_{\rm rel} = -\frac{1}{\tau}\,[\Pi + \pi(Q)], \tag{3}$$

where the first term is issuing from the mean-relaxation-time approximation for the collision integral in the kinetic equation from which Eq.(2) is obtained. As in Ref. 4 we introduce a correction term in $I_{\rm rel}$ which is $\pi(Q)$. The definition of the $\pi(Q)$ function will be given later-on.

Equations (1), (2) are obtained in Ref. 4 on the basis of virial theorems introduced by Chandrasekhar for studies of classical liquid bodies [5] and generalized in [6] for Fermi-systems. Virial theorems represent some weighted integrals over the phase space of one fermion of the many-body equations of motion for the density matrix. One arrives at the former equations making a number of approximations in virial theorems*. Being apparently quite sound in determining the mean characteristics of fusing systems, the approximations lead, however, to serious shortcomings of the theory as it was formulated before.

Staying within the model of Ref. 4 one looses the control on the partition of the energy between different channels. In particular, one has uncertainties in the determination of the part of the energy transmitted to the statistical excitation of intrinsic degrees of freedom. Here we establish a precise expression for the collective energy. We obtain the Hamiltonian function starting from the «macroscopic» equations of motion and establish an exact correspondence between the equations of motion and the energy conservation law.

Another drawback of the theory formulated in Ref. 4 is related with the fact that it is limited to the classical mean values of the quantities involved in the description of the fusion process. Neither thermal no quantum deviations from the trajectory are described in this way. The way to treat thermal fluctuations in elastoplastic systems is shown in Refs. 1, 2, 3 using a schematic version of the model in Ref. 4. The quantal approach to the dissipative phenomena is presented in a number of textbooks (see, e.g., Ref. 7) and is further developed in numerous recent publications (see Ref. 8). Here we extend the ideas presented in the quoted papers for more realistic equations of motion (1), (2) preparing ourselves for the quantitative analysis of effects of fluctuations (thermal and quantal) of macroscopic observables in heavy nuclei collisions.

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*In fact, Eq.(2) in Ref. 4 has slightly different form. Some more careful treatment, than made before, of the corresponding virial theorem leads to the form of this equation given in the present paper.

## 3. Canonical Form of Equations of Motion

We look for an expression for the collective energy corresponding to the equations of motion (1), (2). When the dissipation is absent the collective energy is conserved, and we start with a search for integrals of motion corresponding to Eqs.(1), (2) in the limit when $\tau \to \infty$.

Consider an auxiliary Lagrange function of the following form:

$$L' = \frac{M(Q)}{2} \dot{Q}^2 - U(Q) \tag{4}$$

with

$$\frac{dM(Q)}{dQ} = -4\kappa(Q)\, M(Q), \qquad \frac{dU(Q)}{dQ} = 2W_{2,0}(Q)\, M(Q) \tag{5}$$

and the associated energy function

$$E' = \dot{Q}\frac{\partial L'}{\partial \dot{Q}} - L' = \frac{M(Q)}{2} \dot{Q}^2 + U(Q). \tag{6}$$

In fact, equations (5) have a two-parametric family of solutions: the integration constants remain undefined at this stage. One of them contributes a constant term to the part of the energy given by Eq.(6) and is not important. However, the «mass parameter» $M(Q)$ is determined by (5) only up to a scaling factor ($M_0$). We shall discuss its choice later.

It is easy to check the following relation

$$\frac{dE'}{dt} = M(Q)\, \dot{Q}(\ddot{Q} - 2\kappa(Q)\, \dot{Q}^2 + 2W_{2,0}(Q)). \tag{7}$$

Using Eq.(1) one finds

$$\frac{dE'}{dt} = 2M(Q)\, \dot{Q}\Pi. \tag{8}$$

Now, we introduce a new function

$$\mu(Q) = 2\int_0^Q M(Q')\, dQ' \tag{9}$$

and write the r.h.s. of Eq.(7) in the following form

$$2M(Q)\, \dot{Q}\Pi = \frac{d}{dt}(\mu(Q)\Pi) - \mu(Q)\dot{\Pi} =$$

$$= \frac{d}{dt}(\mu(Q)\Pi) + \mu(Q)\, F_{fs}(Q)\, \dot{Q}. \tag{10}$$

In writing the last equation the limiting form of Eq.(2) at $\tau \to \infty$ is used.

Introducing one more function

$$U_{fs}(Q) = -\int_0^Q dQ'\mu(Q')\, F_{fs}(Q') \tag{11}$$

we arrive at the following equation

$$\frac{d}{dt} I_1 = 0, \tag{12}$$

where

$$I_1 = \frac{M(Q)}{2} \dot{Q}^2 + U(Q) + U_{fs}(Q) - \mu(Q)\,\Pi. \tag{13}$$

In the limit $\tau \to \infty$ Eq.(2) represents another integral of motion:

$$I_2 = \Pi + W_{fs}(Q) = \Pi + \int_0^Q dQ' F_{fs}(Q'). \tag{14}$$

The energy must be a function depending on these two invariants. The $\Pi$ variable is associated with the intrinsic kinetic energy tensor of the fermionic system. Having in mind that the dynamical equations in their present form are linear in $\Pi$ variable and that the kinetic energy must be a positive definite function, we write the following expression for the collective energy:

$$E = I_1 + \frac{C}{2} I_2^2, \tag{15}$$

where $C$ is a constant.

Now, as in Ref. 1 we consider $\Pi$ as a quantity associated with the generalized velocity corresponding to a cyclic variable $Z$. We write:

$$\Pi(t) = \dot{Z}(t) + f(Q). \tag{16}$$

Then the quantity in Eq.(15) takes the form

$$E = \frac{M(Q)}{2} \dot{Q}^2 + \frac{C}{2} \dot{Z}^2 + U_{\mathrm{tot}}(Q) + \xi(Q)\,\dot{Z}, \tag{17}$$

where

$$U_{\mathrm{tot}}(Q) = U(Q) + U_{fs}(Q) - f(Q)\,\mu(Q) + \frac{C}{2}\,[f(Q) + W_{fs}(Q)]^2,$$

$$\xi(Q) = C[f(Q) + W_{fs}(Q)] - \mu(Q). \tag{18}$$

Let us choose $f(Q)$ in such a way that $\xi(Q)$ becomes zero:

$$f(Q) = \frac{\mu(Q)}{C} - W_{fs}(Q). \tag{19}$$

Expression in Eq.(17) is the collective energy of the system. To find the collective Hamiltonian one must introduce into this expression the momenta conjugated to the collective variables $Q$ and $Z$. The momenta are defined by the Lagrangian function, and we turn ourselves to the definition of the latter. We write:

$$L(\dot{Q}, Q, \dot{Z}) = \frac{M(Q)}{2} \dot{Q}^2 + \frac{C}{2} \dot{Z}^2 - U_{\mathrm{tot}}(Q) + \dot{Z}\,L''(Q). \tag{20}$$

Here $L''(Q)$ is a function which must be chosen from the condition that the Lagrange equations

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{Q}_i} - \frac{\partial L}{\partial Q_i} = 0$$

were equivalent to the equations of motion (1) and (2) (here $Q_i$ stands for $Q$ and for $Z$ variables: $Q_1 = Q$, $\dot{Q}_1 = \dot{Q}$, $Q_2 = Z$, $\dot{Q}_2 = \dot{Z}$).

Using Eqs.(16) and (19) in the Lagrange equation

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{Z}} = \frac{d}{dt}(C\dot{Z} + L''(Q)) =$$

$$= \frac{d}{dt}[C(\Pi + W_{fs}) - \mu(Q) + L''(Q)] = 0 \tag{21}$$

and comparing Eq.(21) with Eq.(2), one immediately finds:

$$L''(Q) = \mu(Q). \tag{22}$$

Inserting into Eqs.(18) and (20) the relations (19), (22) and using the earlier introduced definitions one obtains the following expression for the Lagrangian function:

$$L(\dot{Q}, Q, \dot{Z}) = \frac{M(Q)}{2}\dot{Q}^2 + \frac{C}{2}\dot{Z}^2 - U_{\text{tot}}(Q) + \dot{Z}\mu(Q) \tag{23}$$

with

$$U_{\text{tot}} = U(Q) + 2\int_0^Q dQ'M(Q')\, W_{fs}(Q') - \frac{\mu(Q)^2}{2C}\,. \tag{24}$$

The energy conservation is inherent in the Lagrange formalism. On the other hand, it establishes a relation between the generalized velocities and coordinates. The energy defined by Eq.(17) which is the sum of invariants of the motion is conserved in the absence of dissipation. We have found a Lagrangian associated with this energy function leading to the correct form of one of the two equations of motion (of Eq.(2)). This implies that the Lagrange equation in the $Q$ coordinate yields the correct form of Eq.(1) also. The direct calculation confirms this statement.

The term proportional to $1/\tau$ in the r.h.s. of Eq.(1) describes the dissipative phenomena. An appropriate form of canonical equations of motion in the presence of dissipation involves the Rayleigh dissipation function $\Sigma(Q, \dot{q})$ which appears in the Lagrange-Rayleigh equations [9]:

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{Q}_i} - \frac{\partial L}{\partial Q_i} = -\frac{\partial \Sigma}{\partial \dot{Q}_i}\,.$$

The definition of the Rayleigh function does not present any difficulty now. It is:

$$\Sigma = \frac{C}{2\tau}(\Pi + \pi(Q))^2. \tag{25}$$

To end the derivation of the Lagrangian function we must fix the two scaling parameters $M_0$ and $C$. It is possible to do it for the first quantity if the collective mass function is known at a certain point. Thus, we fix the scaling factor in the mass function $M(Q)$ equating it with a given value $M_0$ at $Q = 0$ (see Ref. 4). From Eqs.(17) and (24) one sees that the scaling factor $C$ for the $\Pi$-dependent part of the energy affects the potential term $U_{tot}$ giving the $Q$ dependence of the collective energy. In particular, at small $Q$ values one has:

$$U_{tot} = U(Q) + M_0 Q^2 \left( C - \frac{2M_0}{F_{fs}^0} \right). \tag{26}$$

Assuming that $U(Q)$ contains the totality of the potential energy in vicinity of $Q = 0$, we write:

$$C = \frac{2M_0}{F_{fs}^0}, \tag{27}$$

where $F_{fs}^0 \equiv F_{fs}(Q = 0)$ and $M_0 \equiv M(Q = 0)$.

Finally, one arrives at the following expressions for the Lagrange, energy and Rayleigh functions:

$$L(\dot{Q}, Q, \dot{Z}) = \frac{M(Q)}{2} \dot{Q}^2 - U_{tot}(Q) + \frac{M_0}{F_{fs}^0} \dot{Z}^2 + \dot{Z}\mu(Q), \tag{28}$$

$$E = \frac{M_Q}{2} \dot{Q}^2 + \frac{M_0}{F_{fs}^0} \dot{Z}^2 + U_{tot}, \tag{29}$$

$$\Sigma(Q, \dot{Z}) = \frac{M_0}{\tau} \left[ \left( \dot{Z} + \frac{F_{fs}^0 \mu(Q)}{2M_0} - W_{fs}(Q) \right)^2 + \frac{1}{2} \dot{Z}\pi(Q) \right]. \tag{30}$$

Note, that with the definition of scaling factors made before, the expressions for the collective energy and the other functions become identical with that in Ref. 1 in vicinity of $Q = 0$.

Establishing the form of the collective energy we solve the problem of the heating during the fusion. Indeed, the energy conservation implies that the noncollective (statistical) part of the energy grows with the rate

$$\frac{dE_{stat}}{dt} = \sum_i \frac{\partial \Sigma}{\partial \dot{Q}_i} \frac{\partial H_{coll}}{\partial P_i}, \tag{31}$$

where

$$P_Q = M(Q)\,\dot{Q}, \qquad P_Z = \frac{2M_0}{F_{fs}^0} (\Pi + W_{fs}(Q)).$$

The right-hand side of Eq.(31) is easily calculated giving

$$\frac{dE_{stat}}{dt} = \frac{1}{\tau} \left( \frac{F_{fs}^0}{2M_0} (P_Z - \mu(Q)) + f(Q) + \pi(Q) \right) (P_Z - \mu(Q)). \tag{32}$$

If we fix the function $\pi(Q)$ as

$$\pi(Q) = -f(Q), \tag{33}$$

the expression in Eq.(32) becomes:

$$\frac{dE_{\text{stat}}}{dt} = \frac{F_{fs}^{0}}{2\tau M_0} (P_Z - \mu(Q))^2 \tag{34}$$

so that $dE_{\text{stat}}/dt$ is a nonnegative function of time. Using the relation between the excitation energy and the temperature*, one establishes the time dependence of the latter.

From this point one may start the study of fluctuations in a complete analogy with Refs. 1, 2, 3. When fluctuations in $Q$ are sufficiently small compared to the region of $Q$, where variations of functions entering $L$ are important, one may use the formalism developed in Refs. 2, 3 in a straightforward way. Let $Q(t)$ be the value of $Q$ co-ordinate on the mean trajectory at a time $t$. To study fluctuations one may introduce the simplified expression for the Lagrangian linearizing $L(Q, Q, Z)$ in Eq.(28) in variations of $M(Q)$ and $F_{FS}(Q)$ functions. Then one obtains:

$$L_{\text{appr}} = \frac{M(Q(t))}{2} (\delta\dot{Q})^2 -$$

$$-\left[ U_{\text{tot}}(Q(t)) + \left( \frac{dU_{\text{tot}}(Q)}{dQ} \right)_{Q=Q(t)} \delta Q + \frac{1}{2} \left( \frac{d^2U_{\text{tot}}(Q)}{dQ^2} \right)_{Q=Q(t)} (\delta Q)^2 \right] +$$

$$+ \frac{F_{fs}^{0}\dot{Z}^2}{4M_0} + \dot{Z}\left[ \mu(Q(t)) + \left( \frac{d\mu}{dQ} \right)_{Q(t)} \delta Q \right]. \tag{35}$$

Passing to new variables

$$\delta Q - M(Q(t)) \left( \frac{\partial U}{\partial Q} \right)_{Q=Q(t)}, \qquad \dot{Z} - \frac{2M_0\delta Q}{F_{fs}^{0}} \left( \frac{\partial \mu}{\partial Q} \right)_{Q=Q(t)}$$

one comes back to the Lagrange function of the schematic model of Ref. 3 plus constant terms which play no role in the Lagrangian function and may be neglected. Thus, the formalism developed in Ref. 2 is applicable for the study of fluctuations in a sufficiently large class of systems in which the fluctuations amplitude in $Q$ variable remains always sufficiently small in the scale of variations of the functions $M(Q)$ and $F_{FS}(Q)$.

This accomplishes the formal definition of the elements in the Lagrange–Rayleigh formulation of the fusion dynamics and opens the way to study the thermal fluctuations during the fusion.

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*At the moderate excitation energy the temperature is related with the excitation energy according to equation [10]

$$E_{\text{stat}} = aT^2.$$

## 4. Concluding Remarks

The formal study presented before is necessary to come to quantitative analysis of fluctuations in the fusion process within the model developed in Ref. 4. We note that the existing models do not cope yet with the problem in a satisfactory way. For example, in an important paper on the subject by Fröbrich [11] the necessity of introduction of an arbitrary «cut off» parameter is shown to bring the theory in agreement with experimental data. The cut off parameter is introduced to dismiss the fluctuation effects from the «late» stages of the fusion process appearing in the models with more conventional treatment of dissipation–fluctuation relations. Note, that the qualitative study made in Refs. 2, 3 of effects of fluctuations within the schematic model of Ref. 1 allows one to think that there is no need of such a parameter in the advocated here model of fusion.

In fact, this formalism seems to open many new possibilities of studying the nuclear fusion. We mention two of them: 1) There is no difficulty now to come to the Hamilton formulation of the model which is necessary to study quantal effects; 2) The study makes transparent relations between the model of Ref. 4 and a number of other models of fusion. This, in turn, makes possible to combine positive elements of different approaches to the problem.

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